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BIOGRAPHY.

VASILIEV.

BY GEORGE BRUCE HALSTED.

ALEXANDER VASILIEVITCH VASILIEV was born August 24 (old style), 1853, at Kazan. His father, orientalist already academician, was then Professor of Chinese Literature at the University of Kazan. His mother was a daughter of Simonov, Professor of Astronomy in Lobachévski's time and his predecessor as Rector. In 1855 on the transference of the Oriental Faculty to St. Petersburg, Vasiliev's father removed thither. In 1870 Vasiliev finished the course of the fifth St. Petersburg gymnasium as gold-medalist.

The love for mathematics, awakened in the gymnasium, where in Class VI. he studied Sturm's Differential Calculus, carried him to the mathematical department of the University of St. Petersburg, which then boasted Somov and the great Chebishev (Tchébychev).

As result of his earnest studies for 1870-73 appears the work "On the separation of roots," crowned with a gold medal. In 1874 on his taking his first degree he was invited by the University of Kazan to begin there his teaching as Privat-docent. Though he had planned to continue his studies at Berlin, he accepts this invitation to his birthplace and begins in November, 1874.

His Dissertatio pro venia legendi was entitled "On the separation of the roots of simultaneous equations." In January 1875 he begins to lecture on Functiontheory, all his scholars being older than the professor.

His thesis for the Master's examination, taken in 1878, was "On singular solutions in connection with the new views on the problem of integration of differential equations of first order."

His Master's Dissertation, accepted in May 1880, he prepared abroad, spending the year 1879 in Berlin with Weierstrass and Kronecker, and in Paris with Hermite. His subject was "On the rational functions analogous to the double-periodic." Soon after he was made Docent in the University of Kazan. He spent the next summer in Germany, and wrote "The teaching of mathematics in Berlin and Leipzig Universities."

A question which had so long interested him was treated again in his Doctor's dissertation in 1884, "Theory of the separation of the roots of systems of simultaneous equations." Now chosen Professor Extraordinarius, he was made Professor Ordinarius in 1887.

In 1884 Vasiliev was made president of the physico-mathematical section of the Scientific Society of Kazan University. In 1891 this section changed itself into the independent "Physico-mathematic Society." The eight volumes of Proceedings of this section from 1880 to 1890 contain a series of important articles and criticisms by Vasiliev. Since 1883 he has been the authority on all Russian works in Analysis for the "Fortschritte der Mathematik." In the years 1880-89 Vasiliev was particularly active as member of the local assembly, the Zemstvo, in the government of Kazan. By his influence, the number of folk-schools increased in 1883-89 from 43 to 90, of scholars from 1692 to 3100. Thus his district, Svijaschsk, attained a first rank in all Russia by passing from one scholar for 920 inhabitants to one scholar for 28 inhabitants.

Since 1891 Vasiliev has edited the "Bulletin de la Société Physico-Mathématique de Kasan," which now exchanges with 110 learned publications. In the brilliantly successful celebration of the hundredth birthday of Lobachevski by this society, and the foundation of the Lobachevski Prizes, more than a thousand persons from all over the world took part as subscribers.

The position now held by Vasiliev in the Russian mathematical world may be judged from his being chosen by the Academy of Sciences to report on a great work offered in competition for the Buniakovski Prize. The book received the half prize, while Vasiliev's report is to be honored by insertion in the Transactions of the Academy and the award of the Buniakovski Medal.

The great International Congress of Mathematicians just born into permanent life at its wonderfully successful first meeting, in Zurich, and next to meet at Paris, owes its inception to Vasiliev, who pushed the idea into prominence in every country. It was on his initiative that I brought the matter up in the American Mathematical Society and obtained the signatures of all the members present at the Brooklyn meeting to an endorsement of the idea giving specific credit to Vasiliev as originator. At the actual congress he was most active. From him, Laisant, and G. Cantor emanated the three important resolutions constituting the three commissions of the Congress.

The many works of Vasiliev, being inaccessible because in Russian, will

not be enumerated, but the depth of his thinking and charm of his style may be judged from his great Address on Lobachévski, which it was my good fortune to give to the world in a *literal* translation, not a paraphrase. This translation was greeted by a tremendous outburst of enthusiasm in the mathematical world.

It must here suffice to give a few detached sentences from a mass of letters sent me. "I am astonished to find these researches of such deep philosophical import," writes Professor Daniels of the University of Vermont. "I have read it with intense interest," says Cajori. "This life and work of Lobachevski will be a grand inspiration to mathematicians," says Zerr. "I rejoice that you, 'in the midst of the virgin forests of Texas,' are able to do this work," says Professor Carman. "It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader. Surely no mathematician should miss this gem from farthest Russia," says Dr. L. E. Dickson. "By translating this most interesting Address, you have earned for yourself a title to the thanks of the mathematical world," says Dr. Paul Staeckel, since so well known in this very line. I sent this translation in 1894 to Professor Friedrich Engel of Leipzig, to whom I afterward offered for translation into German my translation of Lobachevski's largest work, "New principles of Geometry with complete theory of parallels." He issued the Address in 1895, saying in his *Nachwort:* "Ich habe die Wassiljefsche Rede nach dem Original uebersetzt, obwohl bereits eine englische Uebersetzung von *G. B. Halsted* (Austin, Texas, 1894) vorlag; es schien mir aber fuer einen Deutschen nicht passend, eine russische Schrift nach einer englishen Uebersetzung zu uebertragen. Selbstverständlich habe ich aber die Halstedsche Uebersetzung ueberall verglichen und bekenne gern, dass sie mir an manchen Stellen gute Dienste geleistet hat."

A French translation and an (incomplete) Spanish translation have since appeared.

This transcendently beautiful production, linking forever the name of Vasiliev with that of Lobachevski, wins both for author and object, the love of every reader.

A personal picture with scene at Kazan the ancient capital of the Tartars, must be reserved for a subsequent chapter: "A Visit to Vasiliev."

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from October Number.]

LVIII. Fig. 29.

$ALMI$ is equivalent to $2IAC = 2BAE$ is equivalent to $ACDE$.

$BKML$ is equivalent to $BKNC$ is equivalent to $BCFH$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

LIX. Fig. 29.
QIK=*RAB*. *BOP*=*AFQ*. *OHKP*=*DEAR*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LX. Fig. 29.
BHK is equivalent to *AFQ*+*DEAR*.
Then *BAIK* is equivalent to *BRQK*.
∴ *ABKI* is equivalent to *ACDE*+*ACFH*.

LXI. Fig. 29.
ABTS is equivalent to 2*ABH* is equivalent to *BCFH*.
STKI=*ALMI* is equivalent to *ACDE*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

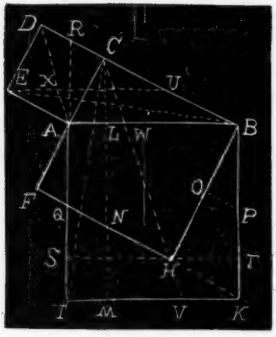


Fig. 29.

LXII. Fig. 29.
Same as in LXI, except that
STKI is equivalent to *ABUE* is equivalent to *ACDE*.

LXIII. Fig. 29.
WBKV, the half of *ABKI*, is equivalent to *BWH*+*BHK*+*HVK*.
But *BHK*=*BCA* is equivalent to *BWC*+*DXE*; and *HVK*=*AXE*.
∴ ½*ABKI* is equivalent to ½*ACDE*+½*BCFH*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LXIV. Fig. 30.
MAF=*NFA*. Then, *KLI*=*FCD*.
ILN=*DEM*. *BHK*=*BCA*.
∴ *ABKI* is equivalent to *ACDE*=*BCFH*.

LXV. Fig. 30.
KHI=*DEF* is equivalent to ½*ACDE*.
HIL=*ALF*.
ILA=*DEF* is equivalent to ½*ACDE*.
BHK=*BCA*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

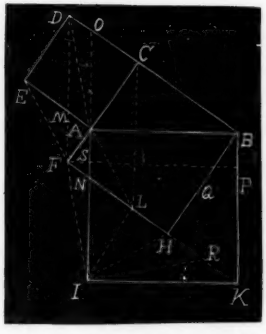


Fig. 30.

LXVI. Fig. 30.
LNOC is equivalent to *LFDC*−*NFDO* is equivalent to *ACDE*.
For *LFDC* is equivalent to *ACDE*+2*FAE*, and 2*FAE* is equivalent to 2*FAD* is equivalent to *NFDO*.
Also, *KLCB* is equivalent to *BCFH*.
∴ *KNOB* is equivalent to *ACDE*+*BCFH*.
But, *ABKI* is equivalent to *KNOB*.
∴ *ABKI* is equivalent to *ACDE*+*BCFH*.

LXVII. Fig. 30.

ISPK is equivalent to $2IFK = 2ADB$ is equivalent to $2ACB + ACDE$ is equivalent to $ACB + FHQ + ACDE$.

SABP is equivalent to *FABQ*.

$\therefore$ *ABKI* is equivalent to $ACDE + BCFH$.

LXVIII. Fig. 30.

$ILR = ACD$, and $ILF = AED$.

Then $IRK = IFA$. $BHK = BCA$.

$\therefore$ *ABKI* is equivalent to $ACDE + BCFH$.

LXIX. Fig. 30.

LNOC is equivalent to $2LAC = 2FED$ is equivalent to *ACDE*.

KLCB is equivalent to *BCFH*.

$\therefore$ *ABKI* is equivalent to *KNOB* is equivalent to $ACDE + BCFH$.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from October Number.]

PROPOSITION XXIX. *Resuming Fig.* 33 *of the preceding proposition: I say every straight AC, which cuts angle BAX, finally at a finite or terminated distance (even in hypothesis of acute angle) will meet BX in a certain point P, if only AC be produced ever more toward the parts of the points X.*

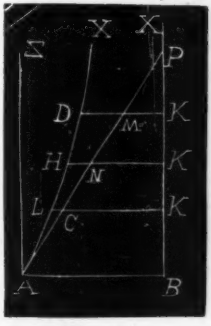


Fig. 33.

Proof. And firstly indeed (lest straight *AC* include space with *AX*) it must meet at finite distance the straights *LK*, *HK*, *DK* in certain points *C*, *N*, *M*; must meet, I say, unless before (and that at a finite distance, just as we maintain) it meets *BX* in some point between the point *B* and one of the points *K*.

Then (from Corollary I. after XXIII.) the angles *ACK*, *ANK*, *AMK* will be obtuse.

Moreover those angles, always obtuse, approach (from the preceding proposition) without any certain limit, to equality with a right angle, when indeed that *AC* is supposed to meet *BX* only at an infinite distance. Therefore such an ordinate *KMD* can be reached that at it the angle *AMK*

exceeds a right angle by less than the angle DAC. But then angle DAC, or DAM, together with angle AMD will be greater than a right angle. Wherefore, the obtuse angle ADM being added, the three angles together of the triangle ADM will be greater than two right angles, which is against the hypothesis of acute angle.

Therefore every straight AC, which cuts that angle BAX, finally at a finite or terminated distance (hypothesis of acute angle) must meet BX in a certain point P. Quod etc.

COROLLARY I. Hence no straight AZ, which toward the parts of the points X makes an acute angle greater than BAX can ever meet BX, either at a finite, or at an infinite distance. For as far as so should happen, now AX dividing angle BAZ, ought (against the premised supposition) to meet BX at a finite distance, as this is demonstrated of the straight AC dividing angle BAX.

COROLLARY II. Moreover it follows that no determinate acute angle will be the maximum of all under which a straight line produced from point A meets BX at finite distance. For if toward the parts of the point X you assume any point higher than the point P, it follows that the straight joining point A with this higher point will make with AB a greater angle than angle BAP. And so ever without any intrinsic end. Wherefore angle BAX (since indeed AX both always approaches to BX, and meets it only at an infinite distance) will be the outside limit of all acute angles under which straights produced from that point A meet the aforesaid BX at a finite distance.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

II.

THE GROUP OF ONE PARAMETER. THE INFINITESIMAL TRANSFORMATION. EXISTENCE OF AN INFINITESIMAL TRANSFORMATION IN A GROUP OF ONE PARAMETER.

5. Consider the plane as a point manifoldness, *i. e*, a space whose space element is the point. The plane will then be two-dimensional,* *i. e* contain ∞^2

*This idea and its bearing in the paragraph are emphasized here not to introduce any unnecessary ultra refinement but because of their use in geometrical illustrations to appear in succeeding articles. For example, the plane is one, two, three, or four dimensional according as, a circle with fixed center, the straight line, a circle of general position, or parabola, be taken as space element, since there are ∞^1, ∞^2, ∞^3, ∞^4, of these elements respectively in the plane. Similarly if the straight line is element it has no dimension, the point has one dimension, the plane two dimensions, and ordinary space four dimensions.

elements, or in other words the position of a point in the plane will be determined by two parameters, the coördinates of the point.

A point-transformation of the plane into itself is an operation by which every point in the plane is conveyed into the position of some point in the same plane. In order to represent this operation analytically, let us take as the coördinate system of reference an ordinary rectangular Cartesian system, x, y; then the point transformation is expressed by two equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \tag{1}$$

where (x, y) is the original point and (x_1, y_1) the transformed point. It is further assumed that the transformation is of such a nature that every point (x_1, y_1) of the plane may be regarded as having originated from some point in the plane by effecting the transformation. This geometrical assumptions finds its analytical condition in the demand that the two functions $\varphi(x, y)$ and $\psi(x, y)$ be independent functions and thus the preceding equations are soluble theoretically with regard to x and y. For, suppose that φ and ψ were not independent, and for example let

$$\varphi = n\alpha(x, y), \quad \psi = m\alpha(x, y);$$

then eliminating $\alpha(x, y)$ from the equations of the transformation

$$x_1 = n\alpha(x, y), \quad y_1 = m\alpha(x, y)$$

we find that the points (x, y) of the plane are transformed into the points (x_1, y_1) of the straight line

$$y_1 = \frac{m}{n} x_1,$$

and hence point of general position no longer is conveyed into point of general position by the transformation.

If the equations (1) be solved with regard to the variables x, y, there result two equations of the form

$$x = \overline{\varphi}(x_1, y_1), \quad y = \overline{\psi}(x_1, y) \tag{2}$$

which represent a transformation that carries the point (x_1, y_1) back into the position (x, y); this transformation (2) is called the *inverse* of the transformation (1). If the transformation (1) be followed by the transformation (2), that is, if the two transformations be carried out successively, we have the two equations

$$x_1 = x, \quad y_1 = y.$$

These equations are very particular cases of equations (1) and hence should represent a transformation. The transformation which they represent obviously transforms a point into itself, or in other words, it leaves all points at rest, for this reason it is called the *identical* transformation.

6. If the equations of a transformation

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a) \tag{3}$$

contain an arbitrary constant a, these equations no longer represent a single transformation but a family of ∞^1 transformations, since the arbitrary parameter a may assume all values from $-\infty$ to $+\infty$. Let us make the hypothesis that the equations (3) represent such a family of transformations that *the successive application of any two transformations of the family is equivalent to a transformation belonging to the same family*; in this case the family (3) is called a *group*; since the group contains *one* parameter a and hence ∞^1 transformations, the group is called a *one parameter group*, or a *group* of one parameter, or symbolically a G_1. Further, since the parameter varies continuously the group is said to be a *continuous* group. As the group contains a finite number of parameters, in this case but one, namely a, it is a *finite continuous group*. The sentence above in italics expresses the group property of the family. A footnote in the preceding article calls attention to the fact that the group property is peculiar to certain classes of families and not common to all of them.

The analytical criterion for a one-parameter group as just defined reveals itself in the following manner. The transformation

$$T_1 \qquad x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a), \tag{4}$$

changes the point (x, y) into the point (x_1, y_1); let T_1 be followed by the transformation T_2 which corresponds to the value a_1 of the parameter and changes the point (x_1, y_1) into the point (x_2, y_2), given by the equations

$$T_2 \qquad x_2=\varphi(x_1, y_1, a_1), \quad y_2=\psi(x_1, y_1, a_1). \tag{5}$$

The transformation, T_3, say, which will carry the point (x, y) directly into the position (x_2, y_2) is found by eliminating (x_1, y_1) from the equations (4) and (5). The elimination yields

$$T_3 \equiv T_1 T_2 \begin{cases} x_2=\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}, \\ y_2=\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\}. \end{cases} \tag{6}$$

If this transformation is to belong to the original family it must be capable of expression in the form

$$x_2=\varphi(x, y, \lambda), \quad y_2=\psi(x, y, \lambda),$$

where λ is a certain function of a and a_1 alone.

Hence the criterion sought is that the two equations

$$\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \varphi\{x, y, \lambda(a, a_1)\},$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \psi\{x, y, \lambda(a, a_1)\},$$

must exist identically for all values of x, y, a, and a_1.

7. In the sequel we shall study only those continuous groups* which contain the *inverse* transformation of every transformation of the group, *i. e.* to a tranformation corresponding to the parameter a,

$$T_1 \qquad x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a),$$

there corresponds a transformation of the family whose parameter is $\bar{a}$ say,

$$T_2 \qquad x_2 = \varphi(x_1, y_1, \bar{a}), \quad y_2 = \psi(x_1, y_1, \bar{a}),$$

such that T_2 cancels T_1 and gives

$$x_2 = x, \quad y_2 = y,$$

the identical transformation.

Accordingly, if the transformations of a group are inverse in pairs the group contains the *identical* transformation. Let a_0 be the value of the parameter which gives the identical transformation, then

$$\varphi(x, y, a_0) \equiv x, \quad \psi(x, y, a_0) \equiv y.$$

A transformation of the family whose parameter is $a_0 + \delta a$, where δa is an indefinitely small quantity will move the point (x, y) through only an infinitesimal distance, such a transformation is called an *infinitesimal transformation*, where by an *infinitesimal* transformation of the group is meant a transformation whose parameter differs by an infinitesimal from that value of the parameter which gives the *identical transformation*.

8. There is a most intimate connection between the notions *infinitesimal transformation* and *one parameter group*. It is proposed to derive now three fundamental theorems of LIE which establish this relationship. The first proves that *every one parameter group contains an infinitesimal transformation*, the second that *every infinitesimal transformation generates a one parameter group*, and the third that *a one parameter group contains but one infinitesimal transformation*.

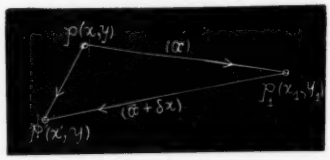
The three theorems show that an infinitesimal transformation may be taken as the representative of a one parameter group.

That a group of one parameter contains an infinitesimal transformation may be seen geometrically in the following manner:

Let a transformation of the group which corresponds to the parameter α and which is designated for convenience by (α) carry the point $p\ (x, y)$ to the position $p_1, (x_1, y_1)$. By assumption the inverse of (α) is contained in the group. Let the parameter of this inverse transformation be $\bar{\alpha}$; $\bar{\alpha}$ is a certain

*The reader must be reminded that this limitation is really not a restriction. LIE has proved in volume III of the Theory of Transformation Groups, Theorem 26, that the defining equations of any continuous group can be derived from those of a group whose transformations are inverse in pairs.

function of α. The transformation $(\bar{\alpha})$ changes all points p_1 into the points p again respectively. A transformation whose parameter differs infinitesimally from $\bar{\alpha}$, say $\bar{\alpha}+\delta\alpha$, will carry the point p_1 not back to p, but to a position at an infinitesimal distance from p, say p'. The successive performance of (α) and $(\bar{\alpha}+\delta\alpha)$ will carry p to p_1 and then to p'; but (α) and $(\bar{\alpha}+\delta\alpha)$ belong to the group, hence the third transformation to which they are equivalent belongs to the group; that is, the transformation which carries p to p', a point infinitesimally near, belongs to the group, or in other words the group contains an infinitesimal transformation.



This geometric process may now be clothed in analytic garb. The first transformation (α) is given by the equations

$$x_1=\varphi(x, y, \alpha), \quad y_1=\psi(x, y, \alpha); \tag{7}$$

the second transformation $(\bar{\alpha}+\delta\alpha)$ by

$$x'=\varphi(x_1, y_1, \bar{\alpha}+\delta\alpha), \quad y'=\psi(x_1, y_1, \bar{\alpha}+\delta\alpha). \tag{8}$$

The elimination of x_1, y_1 from these equations gives the transformation which carries p to p', namely

$$x'=\varphi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}+\delta\alpha\}, \; y'=\psi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}+\delta\alpha\}. \tag{9}$$

Developing* these values in powers of $\delta\alpha$ we have

$$\begin{aligned} x'&=\varphi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}+\frac{\partial\varphi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\dots, \\ y'&=\psi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}+\frac{\partial\psi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\dots. \end{aligned} \tag{10}$$

Now since the transformetions (α) and $(\bar{\alpha})$ are inverse

$$\varphi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}\equiv x, \quad \psi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}\equiv y;$$

hence the equations of the transformation which changes p into p' are

$$\begin{aligned} x'&=x+\frac{\partial\varphi\{\varphi(x, y, \alpha), \psi(x, y, \alpha)\,\bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\dots, \\ y'&=y+\frac{\partial\psi\{\varphi(x, y, \alpha), \psi(x, y, \alpha), \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\dots, \end{aligned} \tag{11}$$

and in this form they represent an infinitesimal transformation since the values

*It is to be remarked once for all that all functions here considered are regular analytic functions and hence expansible by Taylor's Theorem.

of x' and y' differ from x and y respectively by infinitely small quantities. It is easy to see that the coefficients of $\delta\alpha$ do not vanish, for if we put for $\varphi(x, y, \alpha)$ and $\psi(x, y, \alpha)$ their equals x_1 and y_1 respectively, these coefficients equated to zero are

$$\frac{\partial\varphi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv 0, \quad \frac{\partial\psi(x_1, y_1, \bar{\alpha})}{\partial\bar{\alpha}}\equiv 0.$$

But these last identities assert that φ and ψ are free from $\bar{\alpha}$, that is, in general the equations of the group contain no parameter which is contrary to hypothesis.

The quantity $\bar{\alpha}$ is a function of α, since to a transformation $(\bar{\alpha})$ there corresponds, by hypothesis, a completely determinate inverse transformation $(\bar{\alpha})$. The equations (1) of the infinitesimal transformation may be written in the form

$$x'=x+\xi(x, y, \alpha)\delta\alpha+\ldots\ldots, \quad y'=y+\eta(x, y, \alpha)\delta\alpha+\ldots.*$$

LIE thus arrives at the following theorem:

I. *Every one parameter group whose transformations are inverse in pairs contains at least one infinitesimal transformation.*

Princeton University, 22 October, 1897.

(To be Continued.)

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, $1000 each, and the third $457; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1886, that the investment will bring him 8% compound interest?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Mo.

(I). Regarding the notes as bearing simple interest. We get

$\$1000\times 1.24=\$1240$, amount of first note.

$\$1000\times 1.30=\$1300$, amount of second note.

$\$457\times 1.36=\$621.52$, amount of third note.

*These equations contain a constant α which can be arbitrarily chosen, hence we can find an infinitesimal transformation of the group in many different ways. But the sequel will show that all these, excepting a constant factor, are identical in their terms of the first order of infinitesimals.

From September 21, 1886, to April 1, 1888, is $1\frac{1}{3}\frac{9}{6}$ years.
From September 21, 1886, to April 1, 1889, is $2\frac{1}{3}\frac{9}{6}$ years.
From September 21, 1886, to April 1, 1890, is $3\frac{1}{3}\frac{9}{6}$ years.
Let x=amount paid for first note ; y, for second ; z, for third.
$\therefore\ x(1.08)^{1\frac{1}{3}\frac{9}{6}}=1240$, or $\log x=\log 1240-1\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ x=\$1102.448$.
$y(1.08)^{2\frac{1}{3}\frac{9}{6}}=1300$, or $\log y=\log 1300-2\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ y=\$1070.176$.
$\log z=\log 621.52-3\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ z=\$473.743$.
$x+y+z=\$2646.367$=whole amount to be paid for the notes.
(II). If the notes bear compound interest we get,
$\$1000\times(1.06)^4=\$1262.477$, amount of first note.
$\$1000\times(1.06)^5=\$1338.226$, amount of second note.
$\$457\times(1.06)^6=\$648.263$, amount of third note.
$\therefore\ \log x=\log 1262.477-1\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ x=\$1122.43$.
$\log y=\log 1338.226-2\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ y=1101.646$.
$\log z=\log 648.263-3\frac{1}{3}\frac{9}{6}\log 1.08$.
$\therefore\ z=\$494.127$.
$x+y+z=\$2718.20$=whole amount paid for the three notes.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

78. Proposed by J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Required the number of normals that can be drawn from any point (a, b) to the parabola $y^2=2px$.

I. Solution by the PROPOSER.

The equation of the normal to the parabola in terms of its slope, (s), is

$$y=sx-\tfrac{1}{2}(sp)(2+s^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Substituting a, b for x, y in (1), and putting the equation in a new form, we have,

$$s^3+\tfrac{1}{2}p(p-a)s+(2b/p)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Denoting Sturm's functions by F, F_1, F_2, etc., we have the following:

$F = s^3 + (2/p)(p-a)s + (2b/p)$.

$F_1 = 3s^2 + (2/p)(p-a)$.

$F_2 = -2(p-a)s - 3b$.

$F_3 = -b^2 - (8/27p)(p-a)^3$.

Consider five cases.

(1). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically greater than b^2. Sturm's Theorem gives

	F	F_1	F_2	F_3
For $s=+\infty$,	+	+	+	+
$s=-\infty$,	−	+	−	+

Hence the roots are real and unequal.

(2). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically less than b^2. Then

	F	F_1	F_2	F_3
$s=+\infty$,	+	+	+	−
$s=-\infty$,	−	+	−	−

Hence, there is one real root.

(3). Suppose $p-a>0$. Then

	F	F_1	F_2	F_3
$s=+\infty$,	+	+	−	−
$s=-\infty$,	−	+	+	−

Hence, one real root.

(4). Suppose $-b^2-(8/27p)(p-a)^3=0$.

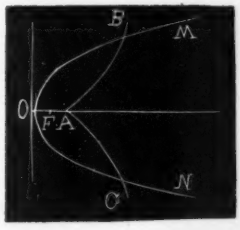
Then there are equal roots, as in this case F and F_1 have a common divisor, and all the roots are real.

(5). Suppose $-b^2-(8/27p)(p-a)^3=0$, and $p=a$.

Then $b=0$, and all the roots are equal, each being 0.

Hence if MON is the given parabola and BAC its evolute, that is, the semi cubical parabola whose equation is

$$b^2 = \frac{8}{27p}(a-p)^3.$$



Then, (1), if the point (a, b) is within (to the right) of the evolute, three normals can be drawn to the parabola; (2), if the point (a, b) is on the evolute, but not at A, two normals can be drawn; (3), if the point (a, b) is A, or is without (to the left) of the evolute, one normal can be drawn to the parabola.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio, and OTTO CLAYTON, Fowler, Ind.

If m be the tangent of the angle which the normal makes with the axis of x, the normal is given by

$$y=mx-pm-(\tfrac{1}{2}p)m^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This passing through (a, b) gives

$$b=am-pm-(\tfrac{1}{2}p)m^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

a cubic in m, showing that the required number is three.

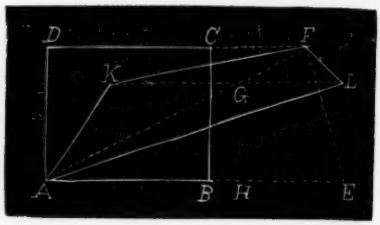
Also solved by *G. B. M. ZERR* and *J. F. LAWRENCE*.

79. Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, University, N. D.

To construct a quadrilateral of given area, the diagonals, one of which is given, cutting each other in given ratios and at a given angle.

I. Solution by JAS. F. LAWRENCE, Freshman Class, Classical Course, Drury College, Springfield, Mo., and the PROPOSER.

Let AC be a rectangle equivalent to the given area and having a side AB equal to one-half of the given diagonal. Produce AB to E, so that $BE=AB$; at A construct $\angle EAF$ equal to the angle to be made by the diagonals, and let AF meet DC produced in F. Divide AF in G in the ratio of division of one diagonal, and AE in H, in the ratio of the given diagonal. On an indefinite line drawn through G parallel to AB lay off GK, CL, equal to AH, HE, respectively, and FK, FL, AK, AL; $AKFL$ is the required quadrilateral.



Join EF. $\triangle FAE=\frown=AC$. It is also equivalent to $AKFL$; for each is equivalent to one-half the parallelogram formed by drawing parallels through the extremities of the diagonals AF, LK.* Hence $AKFL=\frown=AC$; it has also a diagonal $KL=AE=2AB$; its diagonals also are divided in the given ratios, and make an angle $FGL=\angle EAF=$the given angle. Hence, etc.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Lebanon, Va.; OTTO CLAYTON, Fowler, Ind.; and F. R. HONEY, Ph. B., New Haven, Conn.

Let AB be the given diagonal, COB the given angle, $\triangle$ the given area, $m : n$ the given ratio for the known diagonal, $p : q$ the given ratio for the unknown diagonal.

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*From the well known theorem: Any quadrangle is equivalent to one-half the parallelogram formed by drawing lines through its vertices parallel to its diagonals; follow the corollaries—

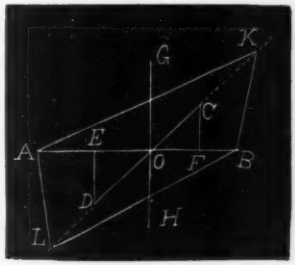
a. Two quadrangles are equivalent if their diagonals are respectively equal and intersect at the same angle. (Triangle a special case.)

b. Any quadrangle is equivalent to the rectangle of its diagonals multiplied by half the sine of their angle.

Let x=unknown diagonal, $\beta=\angle COB$, h=altitude of triangle above AB, h_1=the altitude of the triangle below AB, a=given diagonal.

$$x=2\triangle/a\sin\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Divide AB at O in ratio $m:n$ and draw the indefinite line KL making an angle β with AB.



Let $CO=p$, $DO=q$, $OK=y$, $OL=z$.

$$\therefore\ y=xp/(p+q)=2p\triangle/\{a(p+q)\sin\beta\},$$

$$z=xq/(p+q)=2q\triangle/\{a(p+q)\sin\beta\}.$$

$$\therefore\ h=2p\triangle/\{a(p+q)\}$$

$$h_1=2q\triangle/\{a(p+q)\}.$$

Draw OG, OH perpendicular to AB and $=h$, h_1; draw GK, LH parallel to AB, cutting KL in K and L. $\therefore$ $AKBL$ is the quadrilateral.

III. Solution by A. H. BELL, Hillsboro, Ill., and F. R. HONEY, Ph. B., New Haven, Conn.

Let a, and b, equal the segments of the given diagonals.

Let $x+y$, and $x-y$, equal the segments of the other diagonals.

Let r=the given ratio of the later diagonal, and θ the angle. Then

$$\frac{x+y}{x-y}=r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, $ax\sin\theta+bx\sin\theta=c$ the given area $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$

$$(2)\text{ and }(1)\ x=\frac{c}{(a+b)\sin\theta}=\frac{r+1}{r-1}y \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

$$y=\frac{c(r-1)}{(a+b)(r+1)\sin\theta},\quad x+y=\frac{2cr}{(a+b)(r+1)\sin\theta},\quad x-y=\frac{2c}{(a+b)(r+1)\sin\theta}.$$

Now having all the segments of the two diagonals with the given angle between them, the construction of the required quadrilateral is very simple.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by S. ELMER SLOCUM, Union College, Schenectady, New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the other end and then let go. Show that the chain will run off the pin in about 7-5 second. [*Wright's Mechanics*, page 92.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let 16 feet$=2a$, 9 feet$=b$, and x=the length of the longer part of chain at any time t from the beginning of motion. Then, if m be the mass of a unit of length of the chain, $g=32$, the equation of motion is

$$\frac{d}{dt}\left(2ma\frac{dx}{dt}\right)=2mg(x-a) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiplying both members by $\frac{d(x-a)}{dt}$ and integrating,

$$a\left(\frac{d(x-a)}{dt}\right)^2=g(x-a)^2+C \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When $x=b$, $\frac{d(x-a)}{dt}=0$, and $C=-g(b-a)^2$;

$$\therefore (2) \text{ is } a\frac{d(x-a)}{dt}=g\sqrt{(x-a)^2-(b-a)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{or } dt=\sqrt{\frac{a}{g}}\,\frac{d(x-a)}{\sqrt{(x-a)^2-(b-a)^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\text{Integrating, } t=\sqrt{\frac{a}{g}}\log\left(x-a+\sqrt{(x-a)^2-(b-a)^2}\right)+C' \quad \ldots\ldots (5).$$

When $x=b$, $t=0$; $\therefore C'=\sqrt{\frac{a}{g}}\log(b-a)$, and (5) then becomes

$$t=\sqrt{\frac{a}{g}}\log\left\{\frac{x-a+\sqrt{(x-a)^2-(b-a)^2}}{b-a}\right\} \quad \ldots\ldots\ldots (6).$$

Introducing numbers, $t=1.38$ seconds.

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss., and HENRY HEATON, M. Sc., Atlantic, Iowa.

Let s denote the distance through which the lower end of the string descends in t seconds.

Then, since the acceleration equals the moving force divided by the mass moved,

$$\frac{d^2s}{dt^2}=\frac{2+2s}{16}g.$$

Integrating, $\left(\frac{ds}{dt}\right)^2=\frac{1}{8}g(s^2+2s)$, no constant being added since when

$s=0$, $\frac{ds}{dt}=0$. From the last equation $\frac{ds}{\sqrt{s^2+2s}}=\sqrt{\frac{2g}{4}}dt$ which, by integration, gives $\log_e(s+1+\sqrt{2s+s^2})=\sqrt{\frac{2g}{4}}t$, the constant again being zero, since when $t=0$, $s=0$, and $\log 1=0$.

Taking this between the limits 7 and 0, $t=\frac{7}{5}$, approximately.

Also solved by *G. B. M. ZERR, C. W. M. BLACK, J. SCHEFFER*, and the *PROPOSER*.

53. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas.

Find the locus of the center of gravity of an arc of constant length for a parabola.

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Let u, v be the coördinates of the center of gravity, $y^2=4ax$, be the equation to the parabola for any point on the curve.

$$\therefore su=\int_o^x xds=\tfrac{1}{4}(a+2x)\sqrt{ax+x^2}-\frac{a^2}{8}\log\left(\frac{a+2x+2\sqrt{ax+x^2}}{a}\right)$$

$$=\tfrac{1}{4}(a+2x)\sqrt{ax+x^2}-\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)$$

$$\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)=\frac{a}{4}(s-\sqrt{ax+x^2}).$$

$$\therefore su=\tfrac{1}{2}\sqrt{x(a+x)^3}-\tfrac{1}{4}as \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$sv=\int_o^x yds=\tfrac{4}{3}\sqrt{a(a+x)^3}-\tfrac{4}{3}a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

a and x are both variable in (1) and (2). It does not appear easy to eliminate a and x and thus obtain an equation in u, v.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. Proposed by A. H. BELL, Hillsboro, Illinois.

Given $x^2-114\frac{1}{4}y^2=\mp 3$ to find the least values of x and y in integers.

I. Solution by the PROPOSER.

It can be demonstrated that D, in $x^2-Ay=\pm D$, can be any denominator of the complete quotients from the $\sqrt{A}$, and that x and y are the numerator and denominator of the convergent preceding the term in which D is taken. Now the complete quotients for the $\sqrt{114\frac{1}{4}}$ are

$$\frac{0+\sqrt{114\frac{1}{4}}}{1};\quad \frac{10\frac{1}{2}+\sqrt{114\frac{1}{4}}}{4};\quad \frac{9\frac{1}{2}+\sqrt{114\frac{1}{4}}}{6};\quad \frac{8\frac{1}{2}+\sqrt{114\frac{1}{4}}}{7};\ \text{etc.}$$

No. term ..... 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, etc., reversing.

$\therefore \sqrt{114\frac{1}{4}}=\frac{1}{0},\ \frac{10\frac{1}{2}}{1}$: 5, 3, 2, 1, 2, 1, 6, 2, 1, 1, 10, 10, etc., reversing.

Complete denom'rs=1 : 4, 6, 7, 12, 6, 14, 3, 8, 9, 12, 2, 2, etc., reversing.

Hence x and y are found in the 6th convergent and also the 15th convergent, and x and y=2095 and 196, and also from the 15th term x=42, 307, 834, and y=3, 958, 154. [Also see problem 38.]

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland Maine.

I have not solved this problem as stated, but as I have solved it in this form $x^2-114\frac{1}{4}y^2\pm 3=\square$ .....(1), and as that is a pretty question, I send my solution.

Multiplying by 4, it becomes $4x^2-457y^2\pm 12=\square$ =say $(2x-m^2=4x^2-4mx+m^2)$, from which we find

$$x=\frac{457y^2+m^2\pm 12}{4m}.$$

$(m\pm 12)/4m$ evidently becomes integral when m=6 ; and we have

$$x=\frac{457y^2}{24}+2,\ \text{or}\ \frac{457y^2}{24}+1.$$

$457y^2/24$ becomes integral when $y=12n$, and $x=2742n+2$, or $=2743n+1$, according as the + or − sign before 3 is taken.

If n=1, y=12, and x=2744 or 2743 ; in the former, 3 is negative, and in the latter, positive, in order to make the expression a square.

54. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

I. Solution by the PROPOSER.

$2x^2-2ax+b^2$ is evidently a square when $x=a$. Take $x=y+a$, and substituting, we have $2y^2+2ay+b^2=\square$ =(say)$(my-b)^2$.

Reducing, $y=2(a+bm)/(m^2-2)$. Taking m=2/1, 10/7, 58/41, etc., we have *one* integral series of the value of y, viz : $a+2b$, $49a+10b$, etc. Taking m=3/2, 17/12, 99/70, etc., we have *another* integral series of the value of y, viz:

$8a+12b$, $288a+408b$, etc. By adding a to each term of each series we have two series of the value of x. These series hold good when either a or b is zero ; but if both are zero, $x=0$.

It will be noticed that this solution applies the terms of the question to the expression $2x^2+2ax+b=\square$, the value of x in the latter being a less than in the former.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$2x^2-2ax+b^2=0$. $\therefore x=\frac{1}{2}(a\pm\sqrt{a^2-2b^2})$.

Let $a=p^2+2q^2$, $b=2pq$. Then $x=p^2$ or $2q^2$.

$\therefore$ p	q	a	b	x,
2	1	6	4	4 or 2,
3	2	17	12	9 or 8,
4	3	34	24	16 or 18,
1	2	9	4	1 or 8,
etc.	etc.	etc.	etc.	etc.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step ?

I. Solution by the PROPOSER.

Let O be the center of the circle, A, the end of the first step, and B, the end of the second, and C, the end of the third.

Let $\angle OAB=\theta$, $\angle OBC=\phi$, $OB=x$, and $OC=y$.

Then if the length of the step be taken as the unit of measure, $x=2\sin\frac{1}{2}\theta$, and $y=(x^2+1-2x\cos\phi)^{\frac{1}{2}}=(4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi)^{\frac{1}{2}}$.

If $x=\frac{3}{2}$, B falls upon the circumference of the circle, and $\theta=2\sin^{-1}\frac{3}{4}$. If θ be $>2\sin^{-1}\frac{3}{4}$, and $<\pi$, the second step falls outside the circle. The probability of this is $P_1=(\pi-2\sin^{-1}\frac{3}{4})/\pi$.

If θ be $<2\sin^{-1}\frac{3}{4}$, and $y=\frac{3}{2}$, C falls upon the circumference of the circle, and $4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi=\frac{9}{4}$ or $\phi_1=\cos^{-1}(\sin\frac{1}{2}\theta-5/16\sin\frac{1}{2}\theta)$. Hence if ϕ be $>\phi_1$ the third step falls outside the circle. The chance that ϕ will be $>\phi_1$ and $<\pi$ is $(\pi-\phi_1)\pi$. The chance that θ has any particular value is $d\theta/\pi$. Hence the probability that the third step falls outside the circle is

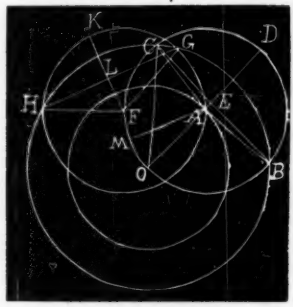
$$P_2 = \frac{1}{\pi^2}\int_0^{2\sin^{-1}\frac{3}{4}} (\pi - \phi_1)\,d\theta.$$

This is not integrable in general terms but its value may be readily approximated by methods of mechanical quadrature.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Let $AO = a$, then $CO = \frac{3}{2}a$.

(1). Let his first step place him at the point A, then in order that he may step outside on the second step he must step somewhere on the arc CDB.



Let $AE = t$, $EC = u$, $\angle DAC = \beta$, P = chance in (1), p = chance in (2).

Now $u^2 = a^2 - t^2 = \frac{9}{4}a^2 - (a+t)^2$, $\therefore\ t = \frac{1}{8}a$.

$\therefore\ \cos\beta = \frac{1}{8}$.

$\therefore\ P = \beta/\pi = \cos^{-1}\frac{1}{8}/\pi = .460106$.

(2). Let chord $OM = \frac{1}{2}a$, then in order that he may step out the third step he must step somewhere on the arc CM or its equal on the opposite side

$$\angle CAM = \delta = \pi - (\beta + OAM)$$

$$= \cos^{-1}\left(\frac{3\sqrt{105} - 7}{64}\right).$$

P_1 = chance he steps on this arc $= \delta/\pi = .379034$.

If his second step places him on arc CM then his third step must place him on the arc GKH. The $\angle KFH$ may vary from 0 to $\cos^{-1}(-\frac{1}{3})$.

$\therefore\ p_1$ = chance that he steps on arc $GKH = \dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi}$.

$\therefore\ p_1 = .304086$.

Now $p = P_1 \times p_1 = \delta/\pi \times \dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi} = .115259$.

Solved with a different result by *CHAS. C. CROSS.*

55. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy ?

Solution by the PROPOSER.

Let p = chance, p_1 = chance that 16th day is clear.

$$\therefore\ p_1 = \frac{\int_0^1 x^{16}\,dx}{\int_0^1 x^{15}\,dx} = \frac{16}{17}. \quad \therefore\ p = 1 - p_1 = \frac{1}{17}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Give a general proof that the centre of gravity, or centroid, determines that point from which the sum of the distances to all other points of a given area is the minimum.

This problem is almost the same as No. 30, Miscellaneous, solutions of which were published on pages 334-5 of Vol. II, and on pages 86-88 of Vol. III. No further solutions have been received. If any of our contributors will attempt other solutions, they will be given in a future number. EDITOR.

50. Proposed by JOHN KEELEY ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.

Describe and compute the actual path traversed by the moon in July and August, 1896, taking into account the motion of the earth around the sun.

No solution of this problem has been received. Dr. S. Hart Wright remarks that "a solution is not possible, as the *actual* path of the moon in space is required, while the moon and the earth describe, in their orbits, neither circles nor ellipses, but curved lines that are *undulatory*, being affected by perturbations due to other planets. If the orbits of the earth and moon were circles or ellipses, the moon's path would be an epicycloidal curve, always concave towards the sun." With the aid of a Nautical Almanac or data of the moon's path during the time asked, it would seem that a practically correct solution of the problem could be effected. We shall be pleased to publish anything further from contributors on this problem. EDITOR.

51. Proposed by F. M. SHIELDS, Coopwood, Miss.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to a city, and bought 156 horses and 177 mules for \$23631; he then traveled farther due north to A, and bought at same price 468 horses and 235 mules for \$52245; he then traveled from A due west 130 miles to B, and bought 120 cows; he then traveled due north to C, and bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 goats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 as much for goats as sheep; at D he turned and traveled in a straight line to the city, a distance equal to the sum of the entire distance he traveled due north from his home H; he sold all his stock at a profit of 20%. How far did he travel from his home H the entire trip around and back to the city? What was the cost of each head of stock, and what was the entire gain?

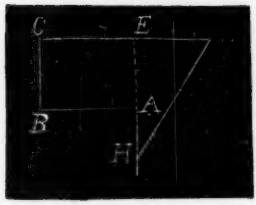
I. Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; CHARLES C. CROSS, Laytonsville, Md.; H. C. WILKES, Skull Run, W. Va.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.

Let x=price of each horse, y=price of each mule.

Then $156x+177y=23631$; and $468x+235y=52245$.

$\therefore\ x=\$80,\ y=\$63.$

$\frac{1}{4}$ of $\$80=\$20$, price of each cow ; $\frac{1}{9}$ of $\$63=\$7$, price of each sheep ; $\frac{1}{2}$ of $\$7=\$3.50$, price of each goat. $120\times 20=2400$; $250\times 7=1750$; $300\times 3.50=1050$.



$\$2400+\$1750+\$1050+\$23631+\$52245$
$=\$81076$, entire cost. 20% of $\$81076=\$16215.20$, entire gain.

Let $AH=u$, $BC=v$.

$\therefore\ (40+u+v)^2=(u+v)^2+(200)^2.$

$\therefore\ u+v=480$ miles.

$\therefore\ 480+40+480+40+330+130=1500$ miles.

II. Solution by **M. A. GRUBER, A. M., War Department, Washington, D. C.**

Draw a diagram of the traveling, and produce line HA to E in line CD. Represent the city by O. Then OED be a right triangle in which $ED=330-130$ $=200$ miles.

Put $a=$distance from home to city. Let $x=OE$; then $OD=x+a$.

Whence $(x+a)^2=x^2+200^2$.

$$\therefore\ x=\frac{200^2-a^2}{2a},\ =\frac{200^2}{2a}-\tfrac{1}{2}a.$$

Now, in order that x may be positive, $\frac{1}{2}a<200^2/2a$; whence $a<200$.

But as the lake is 40 miles wide, a can not be less than 40. Therefore for positive values of x, a may have any value from 40 to 200.

The distance due north $=\dfrac{200^2-a^2}{2a}+a=\dfrac{200^2+a^2}{2a}$; and the entire distance traveled $=\dfrac{200^2+a^2}{a}+460.$

When $a=40$, or if H and O are situated on the lake, the entire distance traveled $=1500$ miles.

When $a=200$, $x=0$, and the city is the farthest north traveled. A would then coincide with O, and C with B.

When $a>200$, x is *negative*. Instead of traveling north from the city, he would then go west from the city to B, and thence *south*, the value of x, to C. For *any* positive value of x, A may be at any point in a due north line between O and E.

Let h, m, c, s, and g be the cost per head, respectively, of horses, mules, cows, sheep, and goats. Then $156h+177m=\$23631$, (1) ; $468h+235m=\$52245$, (2) ; $c=\frac{1}{2}h$, (3) ; $s=\frac{1}{9}m$, (4) ; and $g=\frac{1}{2}s$, (5). From (1) and (2), $h=\$80$, and $m=\$63$. Whence $c=\$20$, $s=\$7$, and $g=\$3\frac{1}{2}$.

$\therefore$ The stock cost $\$23631+\$52245+\$2400+\$1750+\$1050=\$81076$.

By selling his stock at a gain of 20%, he gained $\frac{1}{5}$ of $\$81076=\$16215.20$.

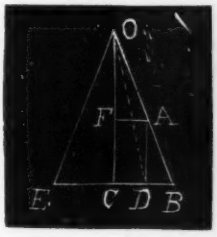
Also solved by *E. W. MORRELL*, and *JOSIAH H. DRUMMOND, LL. D.*

52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Penn.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes $\frac{1}{4}$ of its circumference at the base.

I. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let OBE be section of cone perpendicular to section cutting off segment ADB. By considering projection of hyperbolic section AD on parallel plane through the axis, it is seen that the asymptotes are intersections of latter plane with conical surface. Accordingly, if $OF=a$, $FA=b$, equation to hyperbola is



$$a^2y^2-b^2x^2=-a^2b^2.$$

Now $FA=CD=\frac{1}{2}$ side of square inscribed in circle $EB=\frac{3}{2}\sqrt{2}=b$. $OF:FA=OC:CB$, or $OF:\frac{3}{2}\sqrt{2}=9:3$; $OF=\frac{9}{2}\sqrt{2}=a$. Substituting in formula for area of hyperbola,

$$\text{area } AD=(b/a)x\sqrt{x^2-a^2}-ab\log_e\left(\frac{x+\sqrt{x^2-a^2}}{a}\right),$$

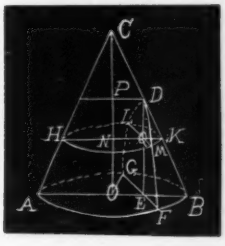
$$=\tfrac{1}{3}\times 9\sqrt{81-\tfrac{81}{2}}-27\log_e\left(\frac{9+\sqrt{81-\frac{81}{2}}}{\frac{9}{2}\sqrt{2}}\right),$$

$$=\tfrac{27}{2}\sqrt{2}-\tfrac{27}{2}\log_e(\sqrt{2}+1).$$

Volume of conical segment $OAD=\frac{1}{3}CD\times$ area AD, $=\frac{27}{2}-\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$.
Area of circular segment $BD=9\pi/4-\frac{9}{2}$, $=\frac{9}{4}(\pi-2)$.
Volume of conical segment $OBD=\frac{1}{3}OC\times$ area DB, $=\frac{27}{4}(\pi-2)$.
Volume $ABD=$ volume $OBD-$ volume $OAD=\frac{27}{4}(\pi-2)-\frac{27}{2}+\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$
$=\frac{27}{4}\pi-27+\frac{27}{4}\sqrt{2}\log_e(\sqrt{2}+1)$, $=2.619+$ cubic inches.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let $AFB-C$ be the cone, $GLDMF$ the section made by the plane cutting off the given segment. Let $AB=6=2R$, $OC=9=h$, $OE=c$. Since $\angle GOF=\pi/2$, $GF=R\sqrt{2}$.



$$\therefore OE=\sqrt{OG^2-GE^2}=\sqrt{R^2-\tfrac{1}{2}R^2}=\tfrac{1}{2}R\sqrt{2}$$
$$=\tfrac{1}{2}(3\sqrt{2})=c.$$

Let $CN=x$, then $CO:OB=CN:NK$.
or $h:R=x:NK$. $\therefore NK=Rx/h=NL$.

Similarly $CP=CO-DF=h-\dfrac{h(R-c)}{R}=\dfrac{hc}{R}$.

Area of segment $LMK=\dfrac{R^2x^2}{h^2}\cos^{-1}\left(\dfrac{ch}{Rx}\right)-\dfrac{c}{h}\sqrt{R^2x^2-h^2c^2}$.

$$\therefore V=\int_{ch/R}^{h}\left\{\frac{R^2x^2}{h^2}\cos^{-1}\left(\frac{ch}{Rx}\right)-\frac{c}{h}\sqrt{R^2x^2-h^2c^2}\right\}dx,$$

$$=\tfrac{1}{3}h\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{R^3-c^2}+\frac{c^3}{R}\log\left(\frac{R+\sqrt{R^2-c^2}}{c}\right)\right\},$$

$$=3\{9\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-9+\frac{9\sqrt{2}}{4}\log(\sqrt{2}+1)\},\ =2.619 \text{ cubic inches.}$$

III. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

I Solution. Designating the radius of the base by r, the altitude by h, and choosing the center of the base for the origin of orthogonal coördinates, CO for the axis of z, the radius OB for the axis of x and a radius parallel to the section FDG for that of y, we find the equation of the cone to be

$$z=(h/r)(r-\sqrt{x^2+y^2},$$

and the volume V of $COFGD$

$$=\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}zdy=\frac{h}{r}\int_0^x dx\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}(r-\sqrt{x^2+y^2})dy,$$

$$=\frac{h}{r}\int_0^x\left(r\sqrt{r^2-x^2}-x^2\log\frac{r+\sqrt{r^2-x^2}}{x}\right)dx,$$

$$=\tfrac{2}{3}hx\sqrt{r^2-x^2}+\tfrac{1}{3}r^2h\sin^{-1}(x/r)-\tfrac{1}{3}\frac{hx^3}{r}\log\frac{r+\sqrt{r^2-x^2}}{x}.$$

Substituting $x=(r/2)\sqrt{2}$, we have for the volume $COFGD$ the expression $\frac{r^2h}{12}[4+\pi-\sqrt{2}.\log(\sqrt{2}+1)]$, and for that of $B-FGD$ $\frac{r^2h}{12}[\pi-4+\sqrt{2}.\log(\sqrt{2}+1)]$

II Solution. Let HK be a circle parallel to AB cutting the hyperbola FDK in the points L and M, and let the diameter HK cut the axis DE at Q. Put $OE=b$, $OF=r$, $CO=h$, $DQ=x$, $LQ=y$. We find from the geometry of the figure $y^2=(2br/x)x+(r^2/h^2)x^2$ as the equation of the hyperbola FDK.

$$\therefore \text{Area of } FDK=2\int dx\sqrt{\frac{2br}{h}x+\frac{r^2}{h^2}x^2} \text{ between the limits } O,$$

and $DE=\frac{(r-b)h}{r}$. Integrating we find for this area the expression,

$$\frac{h}{b}\left(r\sqrt{r^2-b^2}-b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right).$$

$$\therefore \text{Volume of } COFGD=\frac{h}{b}\int_0^b\left(r\sqrt{r^2-b^2}=b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right)db$$

$$= \tfrac{2}{3}hb\sqrt{r^2-b^2} + \tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r} - \tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}\Big),$$

and volume of $BFGD = \tfrac{1}{3}r^2h\cos^{-1}\frac{b}{r} - \tfrac{2}{3}hb\sqrt{r^2-b^2} + \tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}$.

III Solution. Let HK be a circle parallel to AB, and N its centre. Through N draw a diameter parallel to the hyperbolic section FGD. Put $CN=x$, $OE=b$, $BO=r$, $CO=h$, then the area of the circular segment lying between the diameter through N and the parallel chord LM

$$= \frac{r^2x^2}{h^2}\sin^{-1}\frac{bh}{rx} + \frac{br}{h}\sqrt{x^2-\frac{b^2h^2}{r^2}}.$$

$\therefore$ Volume of conical section $COFGD$

$$= \frac{r}{h}\left\{\frac{r}{h}\int x^2\sin^{-1}\frac{bh}{rx} + b\int dx\sqrt{x^2-\frac{b^2h^2}{r^2}}\right\},$$

the integrals to be taken between $h-DE=bh/r$ and h. Thus we find for this volume the expression

$$\tfrac{2}{3}hb\sqrt{r^2-b^2} + \tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r} - \tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r};$$

and for the volume of the conical section $DBFG$,

$$\tfrac{1}{3}r^2h\cos^{-1}\frac{b}{r} - \tfrac{2}{3}hb\sqrt{r^2-b^2} + \tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r}.$$

HISTORICAL NOTE. The famous astronomer KEPLER tried hard to find the volume of such conical sections as the above, but all his efforts proved futile.

Also solved by *GEORGE LILLEY, Ph. D., LL. D.*, and *CHARLES C. CROSS*. Dr. Lilley obtained a numerical result of 6.771 cubic inches, and Professor Cross obtained 2.256979 cubic inches.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

87. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

A and B set out from the same place, and in the same direction. A travels uniformly 18 miles per day, and after 9 days turns and goes back as far as B has traveled during those 9 days; he then turns again, and, pursuing his journey, overtakes B 22½ days after the time they first set out. It is required to find the rate at which B uniformly traveled. [From *Greenleaf's Arithmetic.*]

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest at 6%. Payments: September 1, 1892, $243.50; January 19, 1893, $6.90; April 13, 1894, $19.10; September 19, 1894, $110.90. Amount due February 22, 1897, $229.10.

ALGEBRA.

81. Show that $\dfrac{a_1{}^r}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)}$

$+\dfrac{a_2{}^r}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}+\ldots\dfrac{a_n{}^r}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}$

is zero if r is less than $n-1$; to 1 if $r=n-1$, and to $a_1+a_2+_3+\ldots a_n$ if $r=n$.

[*C. Smith's Treatise on Algebra.*]

82. $\left.\begin{array}{l} y^2+yz+z^2=a^2 \\ z^2+zx+x^2=b^2 \\ x^2+xy+y^2=c^2 \end{array}\right\}$ find x, y, and z.

[*Ibid.*]

GEOMETRY.

83. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

θ being variable, find the locus of a point whose coördinates are

$$a\tan(\theta+\alpha), \quad b\tan(\theta+\beta).$$

84. Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Conn.

Find the locus of a point which will trisect all arcs having a common chord.

85. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Prove by pure geometry. Give direct proof, if possible.

If the bisectors of two angles of a triangle are equal, the triangle is ososceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

MECHANICS.

61. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium. Find how much the body must be depressed, so that when let go, it may just reach the point of suspension.

62. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A particle of mass m moves in the circumference of an ellipse with constant rate v. It is constrained to move in that circumference by attractive forces in the two foci. To determine the magnitude of these forces.

DIOPHANTINE ANALYSIS.

58. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School, Berea, O.

"The base of a right-angled triangle is 105; find all the perpendiculars and hypotenuses to fit it, such that their values shall be integers."

59. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Find the sum of the mth powers of all the numbers less than P and prime to it, and then by substitution find the sume when $m=1, 2, 3, 4, 5$.

NOTES.

THE IRVING HOPKINS FALLACY.

After my having so recently pointed out in THE AMERICAN MATHEMATICAL MONTHLY (Vol. III., pages 122-123) the fallacy of Professor G. C. Edwards of the University of California in his Elements of Geometry in treating parallells, and in *Science* (N. S. Vol. VI. page 491) the gross blunder made by Andrew W. Phillips and Irving Fisher, professors in Yale University, in their Elements of Geometry, could it have been supposed that so respectable a person as Irving Hopkins would deliberately have published the extended fallacy which has just appeared in THE AMERICAN MATHEMATICAL MONTHLY (Vol. IV., pages 251-255) under the ambitious title "Euclidean Geometry without Disputed Axioms"?

It is a simple *petitio principii*. The question is begged in his Proposition IV, which explicitly uses Euclid, III. 31. If any one will turn to III. 31 in any Euclid they will find it proved by Euclid, I. 32. But Euclid I. 32 is the famous angle-sum proposition, which since 1733 has been known to be equivalent to the parallel-postulate, the most disputed of all axioms.

In THE AMERICAN MATHEMATICAL MONTHLY'S serial *Non-Euclidean Geometry*, (Vol. I., page 346) is given the Proposition : In any right-angled triangle the two acute angles remaining are taken together equal to one right angle, in the hypothesis of right angle ; greater than one right angle, in the hypothesis of obtuse angle ; but less in the hypothesis of acute angle. In other words, if the angle inscribed in a semicircle is right the geometry is Euclidean ; if obtuse, Riemannian ; if acute, Lobachevskian. GEORGE BRUCE HALSTED.

Austin, Texas.

There are several errors in Mr. Hopkin's paper on "Euclidean Geometry Without Disputed Axioms," but one is enough to which to call attention. In several places he uses Euclid III., 31, which depends upon I., 32, which depends upon I., 29, which depends upon Axiom 12 !

When will we cease trying to accomplish what the masters have found to be impossible ? BENJ. F. YANNEY.

Mt. Union College, Alliance, Ohio.

NOTE ON DR. LILLEY'S ARTICLE IN THE OCTOBER NUMBER.

There is one statement in Professor Lilley's article in the October number about which I wish to say a few words. Concerning the quotient 0. I define division thus : Having given the product of two factors, and one of the factors, to find the other factor.

Thus, the product of two factors=12,

One of the factors=0,

The other factor=0, (Lilley).

Hence $0\times 0=12$. Do you believe it ?

Take this illustration—

$$12\div 3=4$$
$$12\div 2=6$$
$$12\div 1=12$$
$$12\div .1=120$$
$$12\div .01=1200$$
$$12\div .001=12000$$
$$12\div .0001=120000$$
$$\vdots \qquad \vdots$$
$$12\div -.0001=-120000$$
$$12\div -.001=-12000$$
$$12\div -.01=-1200$$
$$12\div -.1=-120$$
$$12\div -1=-12$$
$$12\div -2=-6$$
$$12\div -3=-4$$

Here the dividend is constant. The divisor varying continuously, suppose, changes sign in passing through zero (absolute) and at the same time the quotient changes sign in passing through infinity.

The following definition of division may assist in reaching a conclusion: Division is the process of finding how many times a number may be subtracted from another without changing the sign of the remainder.

Apply this definition thus: How many times may zero (absolute) be subtracted from 12 without changing the sign of the remainder.

The answer is, an infinity of infinities, rather than zero.

MILTON L. COMSTOCK.

Knox College, Galesburg, Ill.

Upon some of the points about which I shall disagree with Dr. Lilley he can quote in his favor some of the most brilliant mathematicians that the world has produced. Nevertheless I shall endeavor to show that they and he have failed to take a common sense view of the subject. Upon one point I think I am safe in saying that the Doctor's position is unique. The source of his errors lies, in my opinion, in his conception of infinity and zero.

Concerning the former he says: "If $12/⓪=$infinity or *the largest possible number*," etc.

From this I can not but infer that he thinks infinity is a constant and that that constant is the largest possible number. He says "$12/⓪=\ominus$, where $\ominus$ represents no number of times." Again we infer that he believes that while $\ominus$ represents *no number of times*, ∞ must represent *some number of times*.

He uses too many zeros. He has ⓪=absolute zero, ⓪=*no number of times*, and ◎=an infinitesimal. He refers to the latter zero as follows: It is a consequence of confounding the 0 arising from dividing a by infinity, with the absolute zero, that so much confusion has arisen."

He has the authority of Davies and Peck's Mathematical Dictionary for this statement, but this does not make it true. Nothing could be more confusing to the average man of common sense than the Doctor's three zeros.

I have no use for more than one. My mind is perfectly clear as to what that is but it is not so clear as to what ∞ is. It is much easier to tell what ∞ is not than what it is.

If we suppose $a/h=N$, where h is a very small positive quantity, then N is a very large one. As h grows smaller and smaller, N grows larger and larger, but N will *not* become infinite *so long as h has the smallest shadow of value.* So long as h has the slightest value we can form some conception of the value of N. It is only when h becomes equal to 0 that N suddenly swings clear out of our powers of conception. It is then, and then only, that it becomes infinite.

I must dissent from even so great a mathematician as Professor De Morgan when he said that he dated his first clear conception of mathematical infinity from the time when he rejected the relation $a/0=\infty$.

The very fact that he had a clear conception of what he called infinity proved that it was not the real infinity.

I have no criticism to make on Dr. Lilley's disposition of 0/0. I would have liked it better if he had added Art. 175 of his Higher Algebra, which reads: "The symbol 0/0 does not always mean *indetermination.* It is often the result of a particular condition which makes a factor, common to both terms of a fraction become zero. Thus," etc. Here follows the well known illustration by using $\frac{a^2-x^2}{a-x}=a+x$.

He does not find it necessary to introduce the infinitesimal to prove that the expression equals $2a$ when $x=a$, as do many writers on the differential calculus when discussing the expression $\frac{y_1-y}{x_1-x}$. In this he is right, for if $a-x$ were an infinitesimal the value of the fraction would differ from $2a$ by an infinitesimal, and *an equation that differs from the truth by an infinitesimal is not true at all.*

HENRY HEATON.

Atlantic, Iowa.

We see no place for confusion in the use of the symbols 0 and ∞, and, therefore, of course, no necessity of introducing new symbols to avoid confusion. If 0 is a symbol used to denote the absence of quantity, and ∞ to denote a quantity larger than any assignable quantity however large, then all operations with these symbols are meaningless. For example, $5\div\infty$, $0\div5$, $5\div0$, $0\div0$, $0\times\infty$, etc., are impossible operations. Standing apart from conditions imposed upon quantities from which these symbols arise by certain limitations, they have no meaning whatever. Hence, when these symbols do arise in mathematical investigations, they must be interpreted in conformity to fundamental principles and conceptions. When we say that $5\div\infty=0$, we mean that the limit of $5\div$ a quantity which increases indefinitely $=0$, concisely expressed thus $\lim_{h=\infty}\left[\frac{5}{h}\right]=0$.

This is an absolutely accurate statement. 0 is the absolute zero and not an *infinitesimal*. In like manner $5 \div 0 = \infty$ is an abbreviated and inaccurate expression for the following: 5 divided by a quantity which decreases indefinitely gives a quotient larger than any quantity however large, or briefly and *accurately* thus $\lim_{e=0} \left[\frac{5}{e}\right] = \infty$.

Discussion on a subject of this sort is trivial, but if it results in giving clearer notions of the use of 0 and ∞, a good work will have been done.

B. F. F.

BOOKS AND PERIODICALS.

Plane and Solid Analytical Geometry. By Frederick H. Bailey, A. M. (Harvard), and Frederick S. Woods, Ph. D. (Göttengen), Assistant Professors of Mathematics in the Massachusetts Institute of Technology. 8vo. Cloth, 371 pages. Boston and Chicago: Ginn & Co.

Besides the usual subjects treated in the ordinary text-books of Analytical Geometry, the following additional ones are treated with sufficient fullness to give a student a fair knowledge of them, viz: Radical Axis, and Properties of Pole and Polars. More attention should be given to these subjects in the future by the ordinary student. Besides deriving the equations of the conics in the usual way, the authors have also derived the equations by passing a plane through a right circular cone, thus emphasizing the relation of the geometrical to the analytical method of treatment. About seventy pages are given to the treatment of Solid Analytical Geometry. The treatment here is clear and concise, affording the student an excellent introduction to this important subject. B. F. F.

Famous Problems of Elementary Geometry.—The Duplication of the Cube; The Trisection of an Angle; and The Quadrature of the Circle. Authorized translation Vorträge Ueber Ausgewälte Fragen der Elementargeometrie Ausgearbeitet von F. Tägert. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal College. 8vo. Cloth, 80 pages. Price, 55 cents. Boston and Chicago: Ginn & Co.

This book deals with the possibility of elementary geometric constructions in general, the nature of transcendental numbers, and with the transcendence of e and π. While no knowledge of the calculus is needed to read this book, the calculus not being employed in any of the discussions, yet a fair knowledge of the theory of equations and series is absolutely necessary to make it easy reading. The translators deserve the thanks of students and teachers of mathematics, and for putting out books of such scientific value at a very reasonable price, the publishers should receive encouragement by a large sale of this book.

B. F. F.

Popular Scientific Lectures. By Ernst Mach, formerly Professor of Physics in the University of Prague, now Professor of the History and Theory of Inductive Science in the University of Vienna. Translated by Thomas J. McCormack. Second Edition, Revised and Enlarged. 8vo. Cloth, 382 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

These sixteen lectures on various scientific subjects are full of interest to all classes of readers. The lecture "On the Relative Educational Value of the Classics and the Mathematico-Physical Sciences" is especially interesting, and is a fair exposition of the argument *pro* and *con*.

In acquiring an education two things are requisite: first, the development of thought, and second, the power to express thought in a clear and forcible manner. The first is gained by the study of mathematics and the natural sciences, the second, by the classics. Hence, in securing the most symmetrical and stable development of the mind, it is essential that the student pursue his study in the classics, especially Latin, as well as mathematics and the natural sciences. Dr. Mach makes this very pertinent statement: "Here I may count upon assent when I say that mathematics and the natural sciences pursued alone as means of instruction yield a richer education, an education in matter and form, a more general education, an education better adapted to the needs and spirit of the times, than the philological branches pursued alone would yield." In bringing out the translation of these valuable lectures, the translator has the thanks of English readers.

B. F. F.

Field-Manual for Railroad Engineers. By J. C. Nagle, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas. 4½x6¾ inches, Flexible Morocco, xv+394 pages. Price, $2.50. New York: John Wiley & Sons.

This book is in every way a model field-manual. It contains six chapters. Chapter I.—Reconnoisance: Chapter II.—Preliminary Surveys: Chapter III.—Location, Art. 7, Projecting Location; Art. 8, Simple Curves; Art. 9, Compound Curves; Art. 10, Track Problems: Chapter IV.—Transition Curves; Art. 11, Theory of the Transition Curve; Art. 12, Field Work; Art. 13, Transition Curve Problems: Chapter V.—Frogs and Switches; Art. 14, Turnouts; Art. 15, Crossovers; Art. 16, Crossing-Frogs and Crossing-Slips: Chapter VI.—Construction; Art. 17, Definitions, General Consideration, Vertical Curves, Elevation of Outer Rail; Art. 18, Earthworks; Art. 19, Grade and Ballast Stakes, Culverts, Bridges, and Tunnels; Art. 20, Monthly and Final Estimates.

The above abridged outline of the table of contents indicates very imperfectly the scope and character of this work. In it may be found the most essential things to be known in civil engineering discussed in a way not only that may be understood, but that can be easily understood, by any one familiar with algebra, geometry, and trigonometry.

B. F. F.

A Chapter in the History of Mathematics. An Address by Vice President W. W. Beman, Chairman of Section A, before the Section of Mathematics and Astronomy, American Association for the Advancement of Science, Detroit Meeting, August, 1897. Pamphlet, 20 pages.

In this very able address by Professor Beman is gathered together some valuable history concerning the introduction in mathematics of the square root of negative numbers. The address bears evidence of careful research, and is of great interest to all who are concerned about the progress and development of that great body of doctrine known as mathematics.

B. F. F.

Darwin and After Darwin: Part II. Post-Darwinian Questions. Heredity and Utility. 8vo. Cloth, xii and 344 pages. Price, $1.50. With portrait of Romanes. Chicago: The Open Court Publishing Co.

This, as all of Dr. Romanes' works, bears the evident marks of a profound thinker and scholar. The volume before us is chiefly devoted to a consideration of those Post-Darwinian theories which involve fundamental questions of Heredity and Utility, and contains the most valuable results of a deep study of the evolutionary problem. B. F. F.

The Probability of Hit when the Probable Error in Aim is Known with a Comparison of the Probabilities of Hit by the Method of Independent and Parallel Fires from Mortar Batteries. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. Pamphlet, 12 pages. Reprinted from the Journal of the U. S. Artillery, Vol. VIII, No. 2.

The problem considered in this paper is, To find the probability of hit on the target or deck of a ship whose area is $4aA$, where $2a$ is the width of the target in azimuth and $2A$ its length in range, a shot being fired with the intention of hitting the center. B. F. F.

Contributions to the Geometry of the Triangle. By Robert J. Aley, A. M., Professor of Mathematics in the University of Indiana. Pamphlet, 32 pages.

This thesis was accepted by the Department of Mathematics of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy, which is a sufficient testimonial of its importance and value. B. F. F.

Periodico di Mathematica Per L'Insegnamento Secondario. Dott. G. Lazzeri. November-December number.

The Mathematical Gazette. Edited by F. S. Macauley, M. A., D. Sc. October number.

Bollettino della Associazione "Mathesis" Fra Gl'Insignanti di Mathematica delle Scuole Medie.

Revue Semestrielle des Publications Mathomatiques Rédigée sous les auspices de la Société Mathematique d'Amsterdam. Par P. H. Schoute, D. J. Korteweg, J. C. Kluyver, W. Kapteyn, P. Zeeman.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The December number of the *American Monthly Review of Reviews* has several interesting features. Mr. Ernest Knaufft, editor of the *Art Student*, contributes an elaborate study of "John Gilbert and Illustration in the Victorian Era"; Dr. Clifton H. Levy tells "How the Bible Came Down to Us," with a number of reproductions from ancient Biblical manuscripts and printed texts; Lady Henry Somerset pays a tribute to the late Duchess of Teck; an English officer in the Indian service writes about the Ameer of Afghanistan; Mr. E. V. Smalley discusses Canadian reciprocity, and Mr. Alex. D. Anderson summarizes the progress of the American Republics. There is also a 23-page illustrated department devoted to the season's new books, with an introductory chapter, by Albert Shaw, on "Some American Novels and Novelists." Altogether, the *Review* is not lacking in novelty or variety. B. F. F.